2025 : JEE-ADVANCED

1. Given $x_0 \in \mathbb{R}$ such that $\boxed{e^{x_0} + x_0 = 0}$

For given $\alpha \in \mathbb{R}$, define

$$g(x) = \frac{3xe^x + 3x - \alpha e^x - \alpha x}{3(e^x + 1)} \quad ; \forall x \in \mathbb{R}.$$

Here $g(x_0) = \dfrac{3x_0 e^{x_0} + 3x_0 - \alpha e^{x_0} - \alpha x_0}{3(e^{x_0}+1)}$

$$= \frac{3x_0(e^{x_0}+1) - \alpha(e^{x_0} + x_0)}{3(e^{x_0}+1)}$$

$$= x_0 - \frac{\alpha}{3}\left(\frac{e^{x_0} + x_0}{e^{x_0}+1}\right)$$

$\Rightarrow \boxed{g(x_0) = x_0}$; $\boxed{\text{As } e^{x_0} + x_0 = 0}$

$\Rightarrow \boxed{g(x_0) = -e^{x_0}}$

Now ~~to~~ ~~g(x)~~ consider $\displaystyle\lim_{x \to x_0} \left| \frac{g(x) + e^{x_0}}{x - x_0} \right|$

$= \left| \displaystyle\lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} \right|$ (As ~~g(x)~~ ~~is~~ every differentiable functions are continuous)

$= |g'(x_0)|$

Since $g(x) = \dfrac{3xe^x + 3x - \alpha e^x - \alpha x}{3(e^x+1)}$

$$= x - \frac{\alpha}{3}\left(\frac{e^x + x}{e^x+1}\right)$$

$$\Rightarrow g'(x) = 1 - \frac{\alpha}{3}\left[\frac{(e^x+1)^2 - e^x(e^x+x)}{(e^x+1)^2}\right]$$

$$= 1 - \frac{\alpha}{3}\left[1 - \frac{e^x(e^x+x)}{(e^x+1)^2}\right]$$

$$\Rightarrow g'(x_0) = 1 - \frac{\alpha}{3}\left[1 - 0\right] = 1 - \frac{\alpha}{3}$$

$$\Rightarrow |g'(x_0)| = \left|1 - \frac{\alpha}{3}\right|$$

At $\alpha = 2$; $|g'(x_0)| = \frac{1}{3}$;

At $\alpha = 3$; $|g'(x_0)| = 0$;

Hence, option (C).

i.e., For $\alpha = 3$, $\lim\limits_{x \to x_0} \left| \dfrac{g(x) + e^{x_0}}{x - x_0} \right| = 0$.

$\boxed{2.}$ Given area $=\{(x,y)\in \mathbb{R}\times\mathbb{R} : x>0, y>\frac{1}{x}, 5x-4y-1>0; 4x+4y-17<0\}$

Consider

$L_1 : y=\frac{1}{x}$

$L_2 : 5x-4y-1=0$

$L_3 : 4x+4y-17=0$

Solve L_2 and L_3, we get

$9x-18=0 \Rightarrow \boxed{x=2}$

From L_2, $y=\frac{5(2)-1}{4} \Rightarrow \boxed{y=9/4}$ $\therefore A=(2,9/4)$

Solve L_1 and L_2, we get

$5x-4\frac{1}{x}-1=0$

$\Rightarrow 5x^2-x-4=0$

$\Rightarrow 5x^2-5x+4x-4=0$

$\Rightarrow (5x+4)(x-1)=0$

$\Rightarrow \boxed{x=1}$ or $\boxed{x=-4/5}$

From L_1, $\boxed{y=1}$ or $\boxed{y=-5/4}$

$B=(1,1)$ and $B'=(-4/5,-5/4)$

Solve L_1 and L_3, we get

$4x+4\frac{1}{x}-17=0$

$\Rightarrow 4x^2-17x+4=0$

$\Rightarrow 4x^2 - 16x - x + 4 = 0$

$\Rightarrow 4x(x-4) - 1(x-4) = 0$

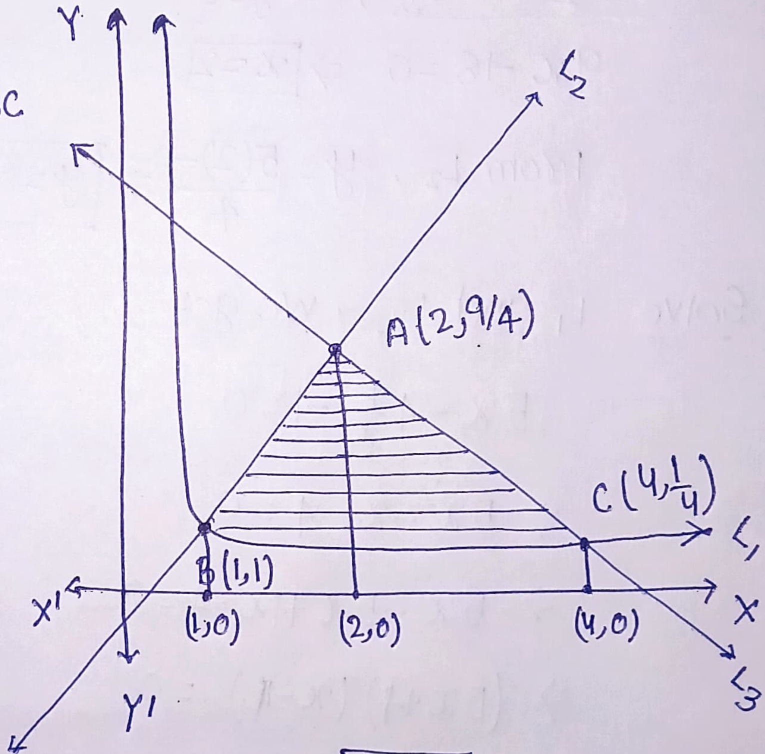
$\Rightarrow (4x-1)(x-4) = 0$

$\Rightarrow x = 1/4 \text{ or } x = 4$

From L_1, $y = 4$ or $y = 1/4$

$\therefore C = (1/4, 4)$ and $C' = (4, 1/4)$

Draw L_1, L_2 and L_3 in XY plane (1^{st} Quadrant)



Area of shaded region ABC

$$= \int_1^2 \left[\left(\frac{5}{4}x - \frac{1}{4}\right) - \frac{1}{x}\right] dx + \int_2^4 \left[\left(\frac{17}{4} - x\right) - \frac{1}{x}\right] dx$$

$$= \left. \frac{5}{8}x^2 - \frac{1}{4}x - \ln x \right|_1^2 + \left. \frac{17}{4}x - \frac{x^2}{2} - \ln x \right|_2^4$$

$$= \frac{5}{8} \times 3 - \frac{1}{4} - \ln 2 + \frac{17}{4} \times 2 - \frac{12}{2} - \ln 2$$

$$= \frac{15}{8} - \frac{1}{4} + \frac{34}{4} - 6 - \ln 4$$

L_1: $\boxed{y = 1/x}$

L_2: $5x - 4y - 1 = 0$

$\Rightarrow \boxed{y = \frac{5}{4}x - \frac{1}{4}}$

L_3: $4x + 4y - 17 = 0$

$\Rightarrow \boxed{y = -x + \frac{17}{4}}$

$$= \frac{15-2+68-48}{8} - \ln 4$$

$$= \frac{33}{8} - \ln 4$$

Hence $ar(ABC) = \frac{33}{8} - \log_e 4$

Option (B)

$\boxed{3.}$ Given $\theta = \tan^{-1}(2\tan\theta) - \frac{1}{2}\sin^{-1}\left(\frac{6\tan\theta}{9+\tan^2\theta}\right)$

where $\sin^{-1}x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\tan^{-1}x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\Rightarrow \sin^{-1}\left(\frac{6\tan\theta}{9+\tan^2\theta}\right) = 2\tan^{-1}(2\tan\theta) - 2\theta$$

$$\Rightarrow \frac{6\tan\theta}{9+\tan^2\theta} = \sin\{2(\tan^{-1}(2\tan\theta)-\theta)\}$$

$$= \frac{2\tan\{\tan^{-1}(2\tan\theta)-\theta\}}{1+\tan^2\{\tan^{-1}(2\tan\theta)-\theta\}}$$

$$= \frac{2\,\dfrac{\tan(\tan^{-1}(2\tan\theta)) - \tan\theta}{1+\tan(\tan^{-1}(2\tan\theta))\cdot\tan\theta}}{1+\left(\dfrac{\tan(\tan^{-1}(2\tan\theta)) - \tan\theta}{1+\tan(\tan^{-1}(2\tan\theta))\cdot\tan\theta}\right)^2}$$

$$= \frac{2\,\dfrac{2\tan\theta - \tan\theta}{1 + \tan\theta\cdot\tan\theta}}{1 + \left(\dfrac{2\tan\theta - \tan\theta}{1 + \tan\theta\cdot\tan\theta}\right)^2} = \frac{2\tan\theta \times (1 + \tan^2\theta)}{\tan^2\theta + (1 + \tan^2\theta)^2}$$

$$\frac{6\tan\theta}{9 + \tan^2\theta} = \frac{2\tan\theta\,(1 + \tan^2\theta)}{\tan^4\theta + 3\tan^2\theta + 1}$$

$\Rightarrow \boxed{\tan\theta = 0}$ or $\dfrac{3}{9 + \tan^2\theta} = \dfrac{1 + \tan^2\theta}{\tan^4\theta + 3\tan^2\theta + 1}$

$\Rightarrow \boxed{\theta = 0}$

$\Rightarrow 3\tan^4\theta + 9\tan^2\theta + 3 = 9 + 10\tan^2\theta + \tan^4\theta$

$\Rightarrow \cancel{4\tan^4\theta + \tan^2\theta - 6 = 0}$

$\Rightarrow 2\tan^4\theta - \tan^2\theta - 6 = 0$

$\Rightarrow 2\tan^4\theta - 4\tan^2\theta + 3\tan^2\theta - 6 = 0$

$\Rightarrow (2\tan^2\theta + 3)(\tan^2\theta - 2) = 0$

$\Rightarrow \tan^2\theta = 2$

$\Rightarrow \tan\theta = \pm\sqrt{2} \Rightarrow \theta = \tan^{-1}(\pm\sqrt{2})$

Hence It has 3 (real) Solutions.

$\therefore$ option (C)

Q4. Given two straight line

$$4x - 3y = 12\alpha \quad \text{———— (i)}$$

$$4\alpha x + 3\alpha y = 12 \quad \text{———— (ii)}$$

$Eq^n(ii) + \alpha\, Eq^n(i)$: $8\alpha x = 12(\alpha^2+1)$

$$\Rightarrow \boxed{x = \frac{3}{2}\left(\frac{1}{\alpha}+\alpha\right)}$$

$Eq^n(ii) - \alpha\, Eq^n(i)$: $6\alpha y = 12(1-\alpha^2)$

$$\Rightarrow \boxed{y = 2\left(\frac{1}{\alpha}-\alpha\right)}$$

$\therefore$ locus of points

$$S = \left\{ \left(\frac{3}{2}\left(\frac{1}{\alpha}+\alpha\right), 2\left(\frac{1}{\alpha}-\alpha\right)\right) : \alpha \in \mathbb{R} - \{0\} \right\}$$

$$\therefore \left(\frac{2x}{3}\right)^2 = \left(\frac{1}{\alpha}+\alpha\right)^2 = \left(\frac{1}{\alpha}-\alpha\right)^2 + 4$$

$$= \left(\frac{y}{2}\right)^2 + 4$$

$$\Rightarrow \frac{x^2}{\frac{9}{4}} - \frac{y^2}{4} = 4 \Rightarrow \boxed{\frac{x^2}{3^2} - \frac{y^2}{4^2} = 1} \quad \text{--- (III)}$$

Slope of tangent (m) = slope of parallel line $4x - \frac{3}{\sqrt{2}}y$

$$\Rightarrow \boxed{m = \frac{4\sqrt{2}}{3}}$$

$$\therefore a^2m^2 - b^2 = 3^2\left(\frac{4\sqrt{2}}{3}\right)^2 - 4^2 = 32 - 16 = 16$$

$$\Rightarrow \boxed{\sqrt{a^2m^2-b^2} = 4}$$

Eq^n of tangent is

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

$$\Rightarrow y = \frac{4\sqrt{2}}{3}x \pm \sqrt{9\left(\frac{4\sqrt{2}}{3}\right)^2 - 16}$$

$$\Rightarrow y = \frac{4\sqrt{2}}{3}x \pm 4$$

$$\Rightarrow \boxed{3y = 4\sqrt{2}x \pm 12} \text{ ——— (iv)}$$

If $x = 0$; $y = \pm 4$

As $q > 0$ so Eq^n of tangent becomes

$$3y = 4\sqrt{2}x + 12$$

$$\Rightarrow \boxed{\frac{y}{4} + \frac{x}{(-3/\sqrt{2})} = 1} \text{ ——— (v)}$$

Eq^n of tangent passing through the points $(p,0)$ and $(0,q)$, $q > 0$

is $$y - 0 = \frac{q-0}{0-p}(x-p)$$

$$\Rightarrow py + qx = pq$$

$$\Rightarrow \boxed{\frac{y}{q} + \frac{x}{p} = 1} \text{ ——— (vi)}$$

From (v) and (vi) are same tangent to.

$$q = 4 \text{ and } p = -3/\sqrt{2}$$

Hence $pq = \frac{-3}{\sqrt{2}} \times 4 = -6\sqrt{2}$.

Q5. Given $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $P = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$; $Q = \begin{pmatrix} x & y \\ z & 4 \end{pmatrix}$; $x, y, z \in \mathbb{R} - \{0\}$

and $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; $a, b, c, d \neq 0$ Such that

$$QR = RP$$

$$\Rightarrow \begin{pmatrix} x & y \\ z & 4 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} ax + cy & bx + dy \\ za + 4c & zb + 4d \end{pmatrix} = \begin{pmatrix} 2a & 3b \\ 2c & 3d \end{pmatrix}$$

$\therefore$ $ax + cy = 2a \Rightarrow a(x-2) + cy = 0 \Rightarrow \boxed{\frac{c}{a} = \frac{2-x}{y}}$ — (i)

$az + c4 = 2c$ — (ii) $\Rightarrow az + 2c = 0 \Rightarrow \boxed{\frac{c}{a} = \frac{-z}{2}}$ — (ii)

~~By cramers ru~~

$\therefore$ ~~x =~~ $\therefore$ From (i) and (ii) $\frac{2-x}{y} = \frac{-z}{2}$

$$\Rightarrow \boxed{4 - 2x + zy = 0} \text{ — (iii)}$$

Again; $bx + dy = 3b \Rightarrow b(x-3) + dy = 0 \Rightarrow \boxed{\frac{d}{b} = \frac{3-x}{y}}$ — (iv)

$zb + 4d = 3d \Rightarrow bz + d = 0 \Rightarrow \boxed{\frac{d}{b} = -z}$ — (v)

From (iv) and (v) $\frac{3-x}{y} = -z$

$$\Rightarrow \boxed{3 - x + yz = 0} \text{ — (vi)}$$

From (iii) and (vi) $4 - 2x = 3 - x$

$$\Rightarrow \boxed{x = 1}$$

From (vi) $\boxed{yz = -2} \Rightarrow \boxed{z = -2/y}$

$\therefore Q = \begin{pmatrix} x & y \\ z & 4 \end{pmatrix} = \begin{pmatrix} 1 & y \\ -2/y & 4 \end{pmatrix}$

Now, $|Q - \lambda I| = \begin{vmatrix} \cancel{x}1-\lambda & y \\ -2/y & 4-\lambda \end{vmatrix}$

$= (1-\lambda)(4-\lambda) + 2$

$= \cancel{2}$

$\Rightarrow \boxed{|Q-\lambda I| = (\lambda-1)(\lambda-4)+2}$

When $\lambda = 2$

$|Q - 2I| = (2-1)(2-4) + 2 = 0$

When $\lambda = 3$

$|Q - 3I| = (3-1)(3-4) + 2 = 0$

When $\lambda = 6$

$|Q - 6I| = (6-1)(6-4) + 2 = 12$

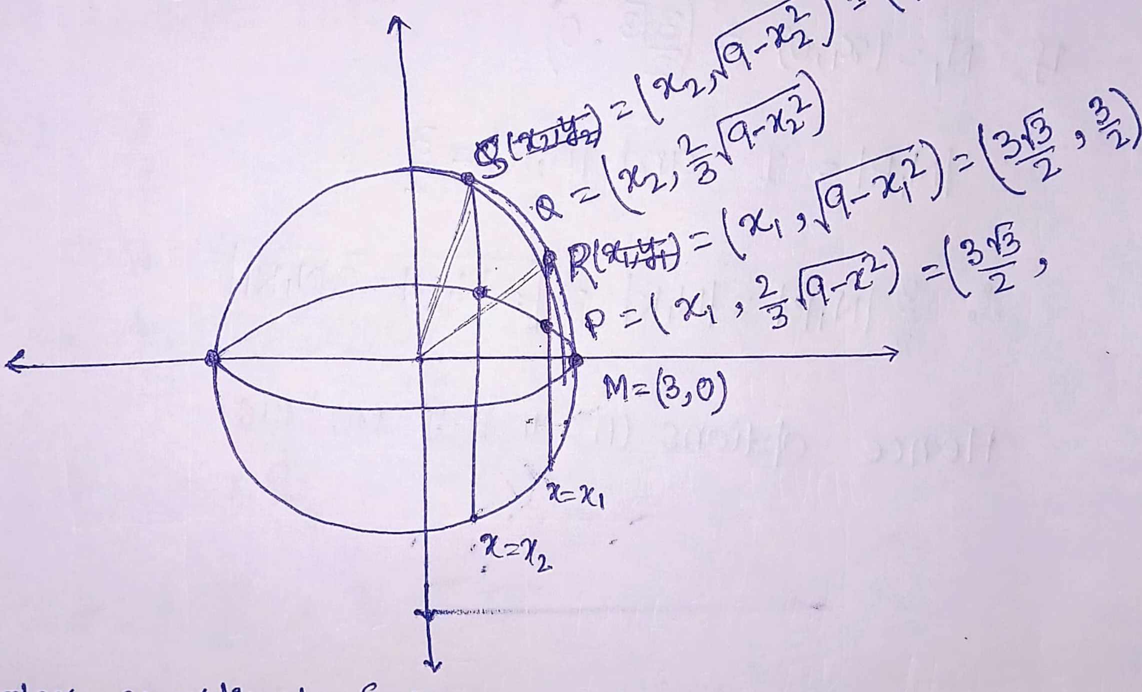
Hence options (A) and (B) are true.

Q7. Given $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two distinct points on the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$, where $y_1, y_2 > 0$.

$C: x^2 + y^2 = 9$

$M = (3, 0)$ and $x = x_1$, $x = x_2$ intersects C at R and S respectively such that ~~their~~ coordinates of their y-axis are positive.

$\angle ROM = \frac{\pi}{6}$ and $\angle SOM = \frac{\pi}{3}$, $O = (0,0)$.



By polar co-ordinate form

Here $R = (x_1, \sqrt{9 - x_1^2}) = \left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$ and $S = (x_2, \sqrt{9 - x_2^2}) = \left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$

$\therefore P = \left(x_1, \frac{2}{3}\sqrt{9 - x_1^2}\right) = \left(\frac{3\sqrt{3}}{2}, 1\right)$

$Q = \left(x_2, \frac{2}{3}\sqrt{9 - x_2^2}\right) = \left(\frac{3}{2}, \sqrt{3}\right)$

Line joining P and Q is: $y - 1 = \frac{\sqrt{3} - 1}{\frac{3}{2} - \frac{3}{2}\sqrt{3}}\left(x - \frac{3\sqrt{3}}{2}\right)$

$= -\frac{2}{3}\left(x - \frac{3}{2}\sqrt{3}\right)$

$\Rightarrow 3y - 3 = -2x + 3\sqrt{3}$

$\Rightarrow \boxed{3y + 2x = 3(1+\sqrt{3})}$

If $N_2 = (x_2, 0) = \left(\frac{3}{2}, 0\right)$

$|N_2Q| = \sqrt{3}$ and $|N_2S| = \frac{3\sqrt{3}}{2}$

$\therefore \frac{3}{2}|N_2Q| = |N_2S| \Rightarrow \boxed{3|N_2Q| = 2|N_2S|}$

If $N_1 = (x_1, 0) = \left(\frac{3\sqrt{3}}{2}, 0\right)$

$|N_1P| = 1$ and $|N_1R| = \frac{3}{2}$

$\therefore \frac{3}{2}|N_1P| = |N_1R| \Rightarrow \boxed{3|N_1P| = 2|N_1R|}$

Hence options (A) and (C) are true.

Q9. Given Differential equation

$$x^2 \frac{dy}{dx} + xy = x^2 + y^2 \; ; \; x > \frac{1}{e} \; ; \; y(1) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2 + y^2 - xy}{x^2}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = 1 + \left(\frac{y}{x}\right)^2 - \left(\frac{y}{x}\right)} \quad \text{———} (i)$$

Here $f(y/x) = 1 + (y/x)^2 - (y/x)$ is a polynomial of degree 2.

Put $\frac{y}{x} = v$

$$\Rightarrow y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow v + x\frac{dv}{dx} = 1 + v^2 - v$$

$$\Rightarrow x\frac{dv}{dx} = (v-1)^2$$

$$\Rightarrow \frac{dv}{(v-1)^2} = \frac{dx}{x}$$

$$\Rightarrow \int \frac{dv}{(v-1)^2} = \int \frac{dx}{x}$$

$$\Rightarrow \frac{-1}{v-1} = \ln x + c$$

$$\Rightarrow v - 1 = \frac{-1}{\ln x + c}$$

$$\Rightarrow v = 1 - \frac{1}{\ln x + c}$$

$$\Rightarrow \boxed{y = x\left(1 - \frac{1}{\ln x + c}\right)} \quad \text{——— (ii)}$$

Apply initial condition $y(1) = 0$

From (ii) $\quad 0 = 1 - \frac{1}{c} \Rightarrow \boxed{c = 1}$

$\therefore$ From (ii) the solution is

$$y(x) = x\left(1 - \frac{1}{\ln x + 1}\right)$$

$$\therefore y(e) = e\left(1 - \frac{1}{2}\right) = \frac{1}{2}e$$

$$y(e^2) = e^2\left(1 - \frac{1}{\ln e^2 + 1}\right)$$

$$= e^2\left(1 - \frac{1}{3}\right) = \frac{2}{3}e^2$$

$$\therefore \frac{2\left(y(e)\right)^2}{y(e^2)} = \frac{2\left(\frac{1}{2}e\right)^2}{\frac{2}{3}e^2} = \frac{3}{4} = 0.75$$

Q12.

Given vectors

$\vec{x} = \hat{i} + 2\hat{j} + 3\hat{k}$; $\vec{y} = 2\hat{i} + 3\hat{j} + \hat{k}$ and $\vec{z} = 3\hat{i} + \hat{j} + 2\hat{k}$

For $\alpha, \beta \in \mathbb{R}^+$ and $\alpha \neq \beta$

$$\vec{X} = \alpha\vec{x} + \beta\vec{y} - \vec{z}$$
$$= (\alpha + 2\beta - 3)\hat{i} + (2\alpha + 3\beta - 1)\hat{j} + (3\alpha + \beta - 2)\hat{k}$$

$$\vec{Y} = \alpha\vec{y} + \beta\vec{z} - \vec{x}$$
$$= (2\alpha + 3\beta - 1)\hat{i} + (3\alpha + \beta - 2)\hat{j} + (\alpha + 2\beta - 3)\hat{k}$$

$$\vec{Z} = \alpha\vec{z} + \beta\vec{x} - \vec{y}$$
$$= (3\alpha + \beta - 2)\hat{i} + (\alpha + 2\beta - 3)\hat{j} + (2\alpha + 3\beta - 1)\hat{k}$$

Since $\vec{X}, \vec{Y}$ and $\vec{Z}$ are lies on a plane

So, $[\vec{X}\ \vec{Y}\ \vec{Z}] = 0$

$$\Rightarrow (\vec{X} \times \vec{Y}) \cdot \vec{Z} = 0$$

$$\Rightarrow \begin{vmatrix} \alpha + 2\beta - 3 & 2\alpha + 3\beta - 1 & 3\alpha + \beta - 2 \\ 2\alpha + 3\beta - 1 & 3\alpha + \beta - 2 & \alpha + 2\beta - 3 \\ 3\alpha + \beta - 2 & \alpha + 2\beta - 3 & 2\alpha + 3\beta - 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 6\alpha + 6\beta - 6 & 2\alpha + 3\beta - 1 & 3\alpha + \beta - 2 \\ 6\alpha + 6\beta - 6 & 3\alpha + \beta - 2 & \alpha + 2\beta - 3 \\ 6\alpha + 6\beta - 6 & \alpha + 2\beta - 3 & 2\alpha + 3\beta - 1 \end{vmatrix} = 0 \quad C_1 \leftarrow C_1 + C_2 + C_3$$

$$\Rightarrow 6(\alpha+\beta-1)\begin{vmatrix} 1 & 2\alpha+3\beta-1 & 3\alpha+\beta-2 \\ 1 & 3\alpha+\beta-2 & \alpha+2\beta-3 \\ 1 & \alpha+2\beta-3 & 2\alpha+3\beta-1 \end{vmatrix} = 0$$

$$\Rightarrow (\alpha+\beta-1)\begin{vmatrix} 0 & -\beta+5 & -\alpha-5\beta \\ 0 & -5\beta+7 & -5\alpha-7\beta \\ 1 & \alpha+2\beta-3 & 2\alpha+3\beta-1 \end{vmatrix} = 0$$

$R_1 \leftarrow R_1+(-2)R_3$
$R_2 \leftarrow R_2+(-3)R_3$

Expansion through 1st column

$$\Rightarrow (\alpha+\beta-1)\begin{vmatrix} -\beta+5 & -\alpha-5\beta \\ -5\beta+7 & -5\alpha-7\beta \end{vmatrix} = 0$$

$$\Rightarrow (\alpha+\beta-1)\begin{vmatrix} -\beta+5 & -\alpha-5\beta \\ -18 & 18\beta \end{vmatrix} = 0$$

$R_2 \leftarrow R_2+(-5)R_1$

$$\Rightarrow 18(\alpha+\beta-1)\left(-\beta^2+5\beta-\alpha-5\beta\right) = 0$$

$$\Rightarrow 18(\alpha+\beta-1)\left(-(\alpha+\beta^2)\right) = 0$$

$$\Rightarrow \alpha+\beta-1 = 0$$

(As if $\alpha+\beta^2=0$ then $\alpha=\beta=0$ but $\alpha,\beta \in \mathbb{R}^+$)

$$\Rightarrow \boxed{\alpha+\beta-3=-2}$$