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TWO DIMENSIONAL CELLULAR AUTOMATA AND ITS REDUCED RULE MATRIX

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Abstract

This paper is mainly focussed on the construction of smaller size matrices for two dimensional Cellular Automata (CA) rules. This reduction in the dimension of rule matrices is small and simple and therefore takes less time to get the successor state of a problem matrix (or CA configuration). Therefore, studying the behavior of CA rules using various properties of matrices are also becomes easy because of its smaller dimension.

Keywords Cellular Automata(CA), Uniform linear rule, Rule matrix, Null Boundary condition, Problem matrix.

1 Introduction

In 2-D rectangular CA as shown in Figure-1 also referred as Moore neighborhood, the next state of a cell is governed by the present status of itself and the states of eight cells lying in its closest proximity. Such relationships among these nine cells are denoted by 512 linear rules and easily reckoned by EX-OR ($\oplus$) operation only [1, 2, 3, and 5].

64	128	256
32	1	2
16	8	4

Fig. 1. Rectangular structure and its nine fundamental rules

The central box represents the cell being considered and all other boxes represent the eight nearest neighbors of that cell. The number within each box represents the rule number characterizing the dependency of the cell on that particular neighbor only. These nine rules are called fundamental rules. In case the cell has dependency on two or more neighboring cells, the rule number will be the arithmetic sum of the numbers of the relevant cells. The number of such linear rules is $2^9 = 512$ which includes rule characterizing no dependency. These rules are denoted by $Rule_0, Rule_1, ..., Rule_{511}$. In [1, 2, 3, and 5] all these rules are characterized by matrices and denoted by M_i i.e., the Rule matrix for $Rule_i$, for $i = 0, 1, 2, 3, ..., 511$. That is if P is a problem matrix of dimension $(m \times n)$, then the dimension of M_i is $(mn \times mn)$. The algorithm used to obtain the successor matrix $P'_{m\times n}$ from the problem matrix $P_{m\times n}$ is the following:

Step 1 : $(P)_{m\times n} \longrightarrow (P_{Col})_{mn\times 1}$
Step 2 : $(M_i)_{mn\times mn}(P_{Col})_{mn\times 1} \longrightarrow (P'_{Col})_{mn\times 1}$
Step 3 : $(P'_{Col})_{mn\times 1} \longrightarrow (P')_{m\times n}$

The above algorithm says that a 2-D problem matrix P arranged in a column vector P_{Col} when multiplied by a rule matrix M_i gives a new column vector P'_{Col} from which the successor state of the CA; P' is obtained. The construction procedure uses (mn) number of basis matrices of size equal to the size of the problem matrix i.e., $(m \times n)$. A linear CA rule is applied to each basis matrix resulting (mn) number of output matrices using which a rule matrix of $(mn \times mn)$ dimension can be constructed. So, with increase in the dimension of a problem matrix the dimension of rule matrix increases. If size of the problem matrix is large then the dimension of the rule matrix becomes large enough to visualize. Because of their incompatible dimensions it is also not feasible to multiply the rule matrix directly to the problem matrix to get the desired outcome. Further studying the behavior of CA rules using various properties of matrices such as rank, nullity, eigen vectors etc. is also difficult due to its large dimension. The above construction algorithm is complicated, time and space consuming. In this paper we have constructed a matrix for $Rule_2$ and using it the matrices for other non-trivial linear CA rules are obtained.

The dimension of $Rule_2$ matrix is much less than $(mn \times mn)$ and therefore it is simple enough for our easy manipulation and also compatible to multiply directly with a problem matrix P giving its successor matrix $P^{'}$.

2 Proposed work

The matrices for the trivial linear rules $Rule_0$ and $Rule_1$ are the null matrix and the identity matrix respectively having dimension same as the dimension of the problem matrix P i.e, $(m \times n)$. The construction methods for finding all 510 linear rule matrices in null boundary condition are shown using some theorems as follows.

Theorem 1 *(Matrix construction for $Rule_2$)*
If $M = (m_{ij})_{n\times n}$ is the binary matrix for $Rule_2$ such that $P_{m\times n}M_{n\times n} = P^{'}_{m\times n}$ then

$$m_{ij} = \begin{cases} 1 & \textit{if } i = j+1 \textit{ and } 1 \leq j \leq (n-1), \\ 0 & \textit{Otherwise.} \end{cases}$$

Proof Let the problem matrix $P_{m\times n} = (p_{ij})_{m\times n}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$, $p_{ij} \varepsilon \{0,1\}$. As $Rule_2$ is the translation of all cells towards left [3] so, when $Rule_2$ is applied to the problem matrix $P_{m\times n}$ then the successor matrix is $P'_{m\times n} = (p^{'}_{ij})_{m\times n}$ where

$$p^{'}_{ij} = \begin{cases} 0 & \text{if } j = n \ \& \ 1 \leq i \leq m, \\ p_{i,j+1} & \text{if } 1 \leq j \leq (n-1) \ \& \ 1 \leq i \leq m. \end{cases} \tag{2.1}$$

Consider a matrix $M = (m_{ij})_{n\times n}$ for $1 \leq j \leq n$ & $1 \leq i \leq n$, when multiplied from the right hand side of $P_{m\times n}$ gives $P'_{m\times n}$.
That is, $P_{m\times n} \times M_{n\times n} = (P^{'})_{m\times n}$. This implies

$$\begin{pmatrix} p_{11} & p_{12} & \dots & p_{1n} \\ p_{21} & p_{22} & \dots & p_{2n} \\ \dots & \dots & \dots & \dots \\ p_{m1} & p_{m2} & \dots & p_{mn} \end{pmatrix}_{m\times n} \begin{pmatrix} m_{11} & \dots & m_{1n} \\ m_{21} & \dots & m_{2n} \\ \dots & \dots & \dots \\ m_{n1} & \dots & m_{nn} \end{pmatrix}_{n\times n} = \begin{pmatrix} p_{12} & \dots & p_{1n} & 0 \\ p_{22} & \dots & p_{2n} & 0 \\ \dots & \dots & \dots & \dots \\ p_{m2} & \dots & p_{mn} & 0 \end{pmatrix}_{m\times n} \tag{2.2}$$

This gives the system of linear equations as:

$$p_{11} \times m_{11} \oplus p_{12} \times m_{21} \oplus \cdots \oplus p_{1n} \times m_{n1} = p_{12} \tag{2.3}$$

$$p_{21} \times m_{11} \oplus p_{22} \times m_{21} \oplus \cdots \oplus p_{2n} \times m_{n1} = p_{22} \tag{2.4}$$

$$\cdots\cdots\cdots$$

$$p_{m1} \times m_{1n} \oplus p_{m2} \times m_{2n} \oplus \cdots \oplus p_{mn} \times m_{nn} = 0 \tag{2.5}$$

The solution of above system of linear equation is

$$m_{ij} = \begin{cases} 1 & \text{if } i = j+1 \;\&\; 1 \leq j \leq (n-1), \\ 0 & \text{Otherwise.} \end{cases}$$

Hence proved.

Theorem 2 *(Matrix construction for other fundamental rules from M)*
The successor matrix of seven fundamental 2-D rectangular CA rules are obtained as follows

1. For $Rule_{32}$, $P_{m\times n} \times M^T_{n\times n} = P'_{m\times n}$
2. For $Rule_8$, $M^T_{m\times m} \times P_{m\times n} = P'_{m\times n}$
3. For $Rule_{128}$, $M_{m\times m} \times P_{m\times n} = P'_{m\times n}$
4. For $Rule_4$, $M^T_{m\times m} \times P_{m\times n} \times M_{n\times n} = P'_{m\times n}$
5. For $Rule_{64}$, $M_{m\times m} \times P_{m\times n} \times M^T_{n\times n} = P'_{m\times n}$
6. For $Rule_{16}$, $M^T_{m\times m} \times P_{m\times n} \times M^T_{n\times n} = P'_{m\times n}$
7. For $Rule_{256}$, $M_{m\times m} \times P_{m\times n} \times M_{n\times n} = P'_{m\times n}$

Proof : Proof of 1, 2, and 3 are similar to ***Theorem 1*** and according to the Parallelogram law of vector addition, $Rule_4$ is the resultant vector of the vectors returned from $Rule_2$ and $Rule_8$. Hence the result is the proof of 4. Similar proofs are for 5, 6, and 7.

Fig-1 can be modified using the ***Theorem 1*** & ***2*** which is shown in Fig-2 as follows.

MPM^T	MP	MPM
PM^T	IP/PI	PM
M^TPM^T	M^TP	M^TPM

Fig. 2. Pictorial representation of *Theorem 1* and *Theorem 2*

Theorem 3 *The successor matrices of 502, 2-D rectangular linear CA rules are obtained by taking EX-OR operation of some or all of the fundamental rule matrices.*

Example 1

For $Rule_3 : P^{'} = PM \oplus PI.$
For $Rule_5 : P^{'} = M^T PM \oplus IP.$
For $Rule_{10} : P^{'} = M^T P \oplus PM.$
For $Rule_{30} : P^{'} = M^T PM^T \oplus M^T P \oplus M^T PM \oplus PM.$

Theorem 4 *(Non-existence theorem)*
1. No matrix exist for $Rule_2$ and $Rule_{32}$ when multiplied from the left side of $P_{m\times n}$ gives its corresponding successor matrix $P^{'}_{m\times n}$.
2. No right hand side matrix exist for $Rule_8$ and $Rule_{128}$ when multiplied with problem matrix $P_{m\times n}$ gives its successor matrix $P^{'}_{m\times n}$.
3. It is impossible to find a single matrix M for $Rule_4$, $Rule_{16}$, $Rule_{64}$ and $Rule_{256}$ such that after multiplication (either left side or right side) with a problem matrix $P_{m\times n}$ gives its corresponding successor matrix $P^{'}_{m\times n}$.

Proof : 1. Let the problem matrix $P_{m\times n} = (p_{ij})_{m\times n}$ for $1 \le i \le m$ and $1 \le j \le n$, $p_{ij} \varepsilon \{0, 1\}$ and ${P'}_{m\times n}$ is the successor state matrix of $P_{m\times n}$ when $Rule_2$ applied once. Assume that there exits a rule matrix $M = (m_{ij})_{m\times m}$ for $1 \le j \le m$ & $1 \le i \le m$. Such that

$$M_{m\times m} \times P_{m\times n} = (P^{'})_{m\times n} \tag{2.6}$$

Equation (2.6) gives a system of linear equations as follows:

$$m_{11} \times p_{11} \oplus m_{12} \times p_{21} \oplus \cdots \oplus m_{m1} \times p_{m1} = p_{12} \tag{2.7}$$

$$m_{21} \times p_{11} \oplus m_{22} \times p_{21} \oplus \cdots \oplus m_{2m} \times p_{m1} = p_{22} \tag{2.8}$$

$$\cdots\cdots\cdots$$

$$m_{m1} \times p_{1n} \oplus m_{m2} \times p_{2n} \oplus \cdots \oplus m_{mm} \times p_{mn} = 0 \tag{2.9}$$

In equation (2.7), there is no term containing p_{12} in the left hand side. So it is impossible to find some value of m_{ij}, $i \leq j \leq m$ in the left hand side of equation (2.7) that gives p_{12}. Similarly for other equations, one will never get values of m_{ij}'s that satisfy the above system linear equations. So, a general matrix does not exist that satisfy the equation (2.6). Hence proved.
Similar proof for $Rule_{32}$, as vector returned from $Rule_{32}$ is opposite to the vector returned from $Rule_2$.
2. Since vectors returned from $Rule_8$ and $Rule_{128}$ are perpendicular to $Rule_2$. Hence, Proof is similar to (1), ***Theorem 4***.
3. [By method of construction] Let $P_{m\times n}$ be a problem matrix and we need to find M such that it gives its successor matrix $P'_{m\times n}$ after $Rule_4$ is applied on $P_{m\times n}$. That is

$$M_{m\times m} \times P_{m\times n} = P'_{m\times n} \tag{2.10}$$

which implies

$$m_{11} \times p_{11} \oplus m_{12} \times p_{21} \oplus \cdots \oplus m_{1m} \times p_{m1} = p_{22} \tag{2.11}$$

$$\cdots\cdots\cdots$$

$$m_{m1} \times p_{1n} \oplus m_{m2} \times p_{2n} \oplus \cdots \oplus m_{mm} \times p_{mn} = 0 \tag{2.12}$$

Since p_{22} is not at all present in the left hand side of equation (2.11). So, we can't get p_{22} from equation (2.11). Similarly for other equations also we can't get their corresponding p'_{ij} elements.
Similarly, if we multiply M with P on right hand side, we also get a set of linear equations having no solutions. Hence it is impossible to construct a matrix M that satisfy equation (2.10).
Since, $Rule_{16}$, $Rule_{64}$ and $Rule_{256}$ behave similar to $Rule_4$. Similar proof also applies to these rules.

3 Efficiency

If the problem matrix is $P_{m\times n}$ whose successor state matrix is $P'_{m\times n}$ after $Rule_i$ is applied once i.e., $P_{m\times n} \xrightarrow{Rule_i} P'_{m\times n}$.

According to the Rule Matrix theory used earlier [1, 2], the dimension of rule matrix M_i is $(mn \times mn)$ and the problem and its successor output matrix have dimension $(mn \times 1)$. And they are related to each other by the following equation

$$M_{mn\times mn} \times P_{mn\times 1} = P'_{mn\times 1} \tag{3.1}$$

Thus using earlier theory, the space and the time complexities to get the successor state of a problem matrix $P_{m\times n}$ is shown in Table 4.1.

Table 4.1

	Space-complexity	Time-complexity	Rectangular CA rules
Earlier work Ref. [1, 2]	$O(m^2n^2)$	$O(m^2n^2 + 2mn)$	All 512 rules

But using the construction procedure discussed in **Section 2**, a matrix is multiplied with the problem matrix from the left side or from the right side or from both sides to get successor state matrix. Hence both the space and time complexities in this case is different for different CA rules as shown in Table 4.2.

Table 4.2

	Space Complexity	Time Complexity	Rectangular CA rules
Left side	$O(n^2)$	$O(mn^2)$	0, 1, 2, 3, 32, 33, 34 & 35
Right side	$O(m^2)$	$O(m^2n)$	0, 1, 8, 9, 128, 129, 136 & 137
Both side	$O(m^2 + n^2)$	$O(mn(m + n))$	All other 498 rules

Comparing above tables, one can observe that both the space and time complexities are very less in Table 4.2 than Table 4.1. For a square matrix (i.e., when m = n) both the space and the time complexity is $O(n^4)$ in Table 4.1 but in this work (Table 4.2) the space complexity is $O(n^2)$ and the time complexity is $O(n^3)$. Since the problem matrix may be in any dimension, so our proposed theory is much efficient than the earlier work [1, 2, and 5].

4 Conclusion

In this paper, the matrix for $Rule_2$ is constructed whose dimension is same as that of the problem matrix and using only this matrix with the help of an identity matrix, the output of all other 511 2-D uniform linear CA rules in nine-neighborhood are obtained. The efficiency of the proposed method to find the successor state of a CA rule is compared with an earlier work and found better.

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